Strain Energy Methods

Worked Example 1 – Combined Strain Energy Case

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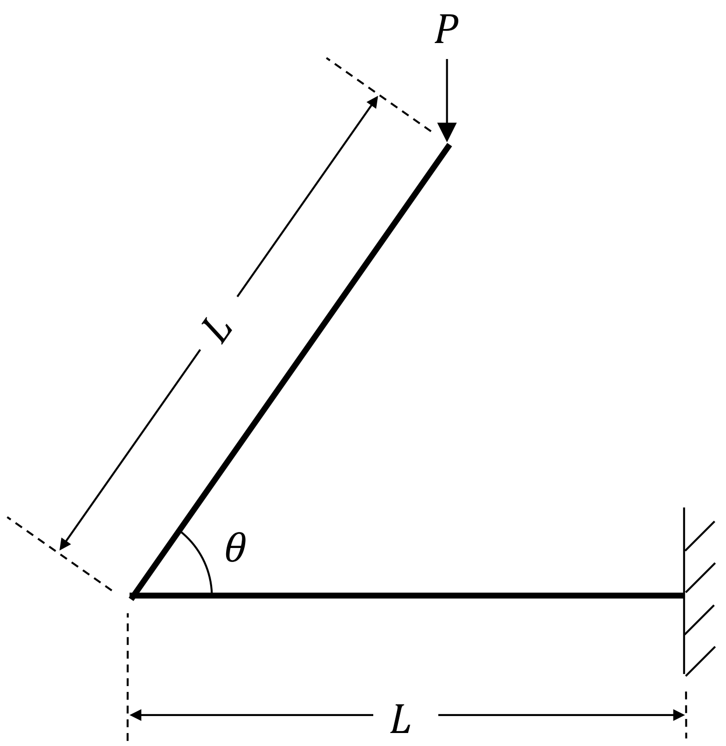
Worked Example 1

Built-in Angled Beam Subjected to a Load at the Free End

The bent uniform bar, shown below, has a circular cross-section of $40\ \mathrm{mm}$ diameter and is subjected to a vertical load, P, of $16\ \mathrm{kN}$ at one end and is clamped at the other.

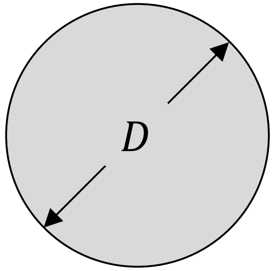


Problem

Use strain energy to determine the vertical deflection at the position of the applied load.

Assume $E = 225\ \mathrm{GPa}$, $L = 0.75\ \mathrm{m}$ and $\theta = 55°$.

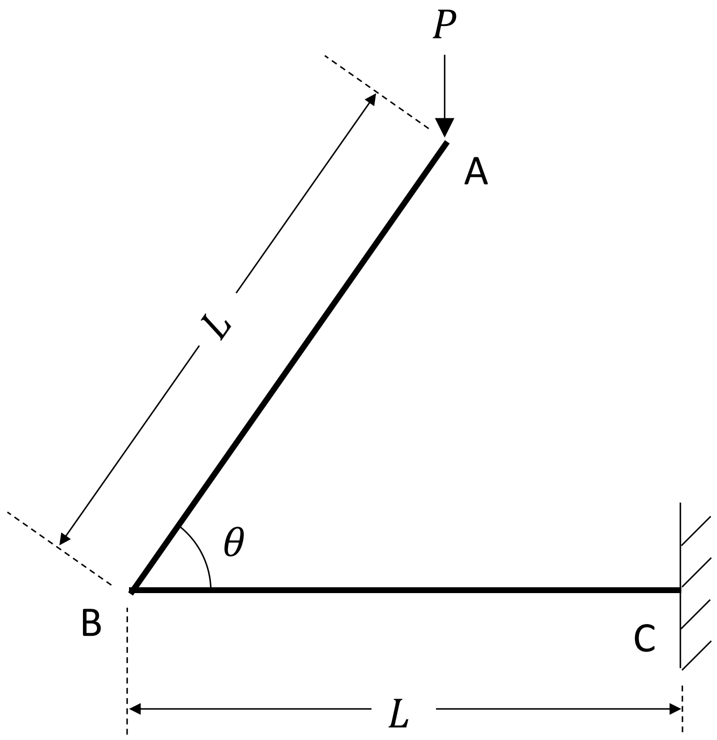
Second Moment of Area, I, calculation



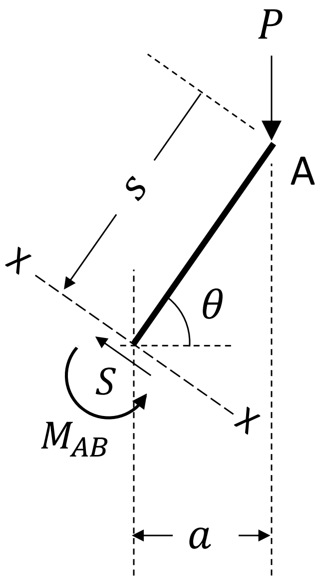
Beam cross-section

$$\therefore I = \frac{\pi D^4}{64} = \frac{\pi \times 40^4}{64} = 125{,}663.71\ \mathrm{mm}^4$$

Labelling the structure



Section AB (*bending only*)



Free body diagram

Taking moments about X-X:

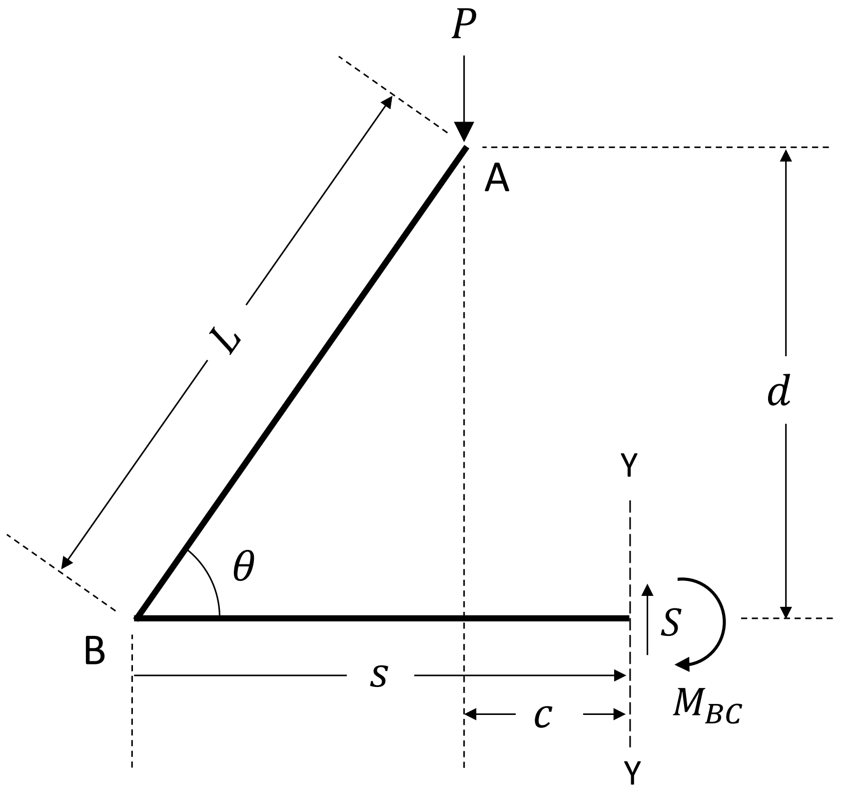
$$M_{AB} = Pa = Ps\cos\theta$$

Substituting this into the equation for strain energy in a beam under bending gives,

$$U_{AB} = \int \frac{{M_{AB}}^2}{2EI}\mathrm{d}s = \int_0^L \frac{(Ps\cos\theta)^2}{2EI}\mathrm{d}s = \frac{(P\cos\theta)^2}{2EI}\int_0^L s^2\mathrm{d}s = \frac{(P\cos\theta)^2}{2EI}\left[\frac{s^3}{3}\right]_0^L$$

$$\therefore \boldsymbol{U_{AB} = \frac{P^2L^3}{6EI}\cos^2\theta}$$

Section BC (*bending only*)



Free body diagram

Taking moments about Y-Y:

$$M_{BC} = Pc = P(s - L\cos\theta)$$

Substituting this into the equation for strain energy in a beam under bending gives,

$$U_{BC} = \int \frac{{M_{BC}}^2}{2EI}\mathrm{d}s = \int_0^L \frac{\left(P(s - L\cos\theta)\right)^2}{2EI}\mathrm{d}s$$

$$= \frac{P^2}{2EI}\int_0^L (s^2 - 2Ls\cos\theta + L^2\cos^2\theta)\mathrm{d}s = \frac{P^2}{2EI}\left[\frac{s^3}{3} - Ls^2\cos\theta + L^2 s\cos^2\theta\right]_0^L$$

$$\therefore \boldsymbol{U_{BC} = \frac{P^2L^3}{2EI}\left(\frac{1}{3} - \cos\theta + \cos^2\theta\right)}$$

Deflection at the Tip of the Beam

Total Strain Energy:

$$U = U_{AB} + U_{BC} = \frac{P^2L^3}{6EI}\cos^2\theta + \frac{P^2L^3}{2EI}\left(\frac{1}{3} - \cos\theta + \cos^2\theta\right)$$

$$\therefore U = \frac{P^2L^3}{2EI}\left(\frac{4\cos^2\theta}{3} - \cos\theta + \frac{1}{3}\right)$$

Differentiating this with respect to the applied load, P, in order to calculate vertical deflection at position A, u_{v_A}:

$$u_{v_A} = \frac{\partial U}{\partial P} = \frac{PL^3}{EI}\left(\frac{4\cos^2\theta}{3} - \cos\theta + \frac{1}{3}\right)$$

Substituting values for P, L, E, I and θ into this gives:

$$\boldsymbol{u_{v_A} = 47.36\ \mathrm{mm}}$$